Qualifying Exam (May 2018): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.

Do 2 out of problems 1,2,3.
Do 2 out of problems 4,5,6.
Do 3 out of problems 7,8,9,10,11,12.

All problems are weighted equally. **On this cover page write which seven problems you want graded.**

problems to be graded:

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Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

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Signature

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(1). The following 2 parts are independent of each other.
(a). Consider an LP that had a variable x_i unrestricted in sign. In converting the problem to standard form, x_i was replaced by a pair of nonnegative variables: $x_i = x_i' - x_i''$, $x_i', x_i'' \geq 0$. Can a BFS to this LP include both x_i' and x_i'' as basic variables? Explain briefly!
(b). Consider a maximization linear programming problem in which the current BFS is non-degenerate. Suppose that x_k is the only variable among the non-basic variables in our current tableau for which $z_k - c_k < 0$ ($z_j - c_j \geq 0$ for $j \neq k$). Suppose you know that the LP has a finite optimal solution. Show that any optimal solution to this LP must have $x_k > 0$.

(2). Consider an LP with upper bounds: $\min\{cx \mid Ax = b,\ 0 \leq x \leq u\}$. A basic feasible solution is said to be *degenerate* if one or more of the basic variables is equal to its upper or lower bound. Assume all BFS's for the LP are non-degenerate for all three parts:
(a). Prove that if a non-basic variable x_k at its lower bound with $z_k - c_k > 0$ is increased, then the new $z <$ old z.
(b). Prove that if a non-basic variable x_k at its upper bound with $z_k - c_k < 0$ is decreased, then the new $z <$ old z.
(c). Prove that the bounded-variable simplex method terminates in a finite number of iterations.

(3). Consider the following resource allocation problem and the accompanying optimal tableau (x_5, x_6, x_7 are the respective slack variables):

$$\begin{aligned} \max \quad & z = 15x_1 + 8x_2 + 10x_3 + 12x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_4 \leq 20 \\ & x_1 + x_2 + x_3 + x_4 \leq 54 \\ & 2x_1 + x_3 + x_4 \leq 36 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	9	0	0	2	4	0	10	440
x_2	0	1/2	1	0	1/2	1/2	0	0	10
x_6	0	-3/2	0	0	-1/2	-1/2	1	-1	8
x_3	0	2	0	1	1	0	0	1	36

(a). What are the shadow prices of the resources (constraints)? If you were to choose between increasing the amount of resource 1, 2, or 3 by 1 unit, which would you choose to increase and why?
(b). Suppose the coefficient of x_4 in the objective function changes from 12 to 16. Use sensitivity analysis to find the new optimal solution.
(c). Suppose that the available amount of resource 1 changes from 20 to 40. Use sensitivity analysis to find the new optimal solution.
(d). Suppose that constraint $3x_1 + 2x_2 + 2x_3 + x_4 \leq 80$ is added to the problem. Use sensitivity analysis to find the new optimal solution.
(e). Suppose that a new product is proposed with objective coefficient 16 and consumption vector $a_8 = (1, 2, 1,)^t$. Use sensitivity analysis to find the new optimal solution.

(4). Let $\{N(t), t \geq 0\}$ be a Poisson process with arrival rate λ. Let $\{A(t), t \geq 0\}$ be the age process associated with $\{N(t)\}$, i.e., $A(t) = t - S_{N(t)}$, where S_n is the time of the occurrence of the nth event. For a given $t > 0$, compute $E[A(t)]$.

(5). Consider an $M/M/1/K$ queue with Poisson arrival $PP(\lambda)$ and i.i.d. $\exp(\mu)$ service times. Let p_K be the long run probability that there are K customers in system (note that an arriving customer seeing K customers in system is lost). Use the mean value approach (i.e., by combining Little's law with the PASTA property) to find L, the long run expected number of customers in system (your final answer should only contain K, λ, μ, and p_K).

(6). Consider two single-server stations in series with Poisson arrivals at rate λ at the first station and exponential service time at rate μ_i at station i, $i = 1, 2$. Assume that no queue is allowed at either station. Both stations may operate simultaneously except that if a customer completes service at station 1 when station 2 is busy, station 1 is blocked from accepting another customer. The blocking customer will not begin service at station 2 until the customer at station 2 departs. The objective is to find the long run average number of customers in system.

(i) Define states and draw a transition rate diagram.

(ii) Give the balance equations (do not solve them).

(iii) In terms of the long run probabilities (i.e., assuming the solution to the balance equations in (ii) is known), find the fraction of arrivals lost.

(iv) Assuming the long run probabilities are known, find the long run average number of customers in system.

(7). Let $S = \{s_1, \ldots, s_n\}$ be a set of n line segments in the plane, in general position (no 3 endpoints are collinear), with no horizontal segments and no vertical segments.

(a). How efficiently (in big-Oh) can one determine if there are any crossing points among the segments S? Justify briefly. (You need not give any algorithmic details.)

(b). We say that S is in *convex position* if every segment s_i appears as an edge of the convex hull of S. How efficiently (in big-Oh) can one determine if S is in convex position? Give the best bound you can and explain briefly.

(c). How efficiently (in big-Oh) can we determine if the convex hull of S is a hexagon? Explain briefly.

(d). For each s_i, let h_i denote the halfplane of points that are on or to the left of the line ℓ_i that contains s_i. (Thus, the points $p \in h_i$ are exactly those points in the plane for which the rightwards (horizontal) ray from p intersects the line ℓ_i.) Let $Q = h_1 \cap h_2 \cap \cdots \cap h_n$ be the intersection set of these n halfplanes. How efficiently can one decide whether or not the set Q is empty? Explain briefly.

(e). For each s_i, let h_i denote the halfplane of points that are on or to the left of the line that contains s_i. Let $Q = h_1 \cap h_2 \cap \cdots \cap h_n$ be the intersection set of these n halfplanes. How efficiently can one determine the rightmost point of the set Q? Explain briefly.

(f). Suppose we want to preprocess S for the following type of query very efficiently: Given a query point q, does q see the origin (point (0,0))? (i.e., does the line segment from the origin to point q intersect any of the segments S?) What preprocessing/space/query time can you achieve for this? Explain briefly.

(g). Suppose we want to preprocess S for the following type of query very efficiently: Given two query points q_1 and q_2 ($q_2 \neq q_1$), does the line through q_1 and q_2 intersect any of the segments S? What preprocessing/space/query time can you achieve for this? Explain briefly.

(8). Let P be a simple n-gon in the plane.

(a). Suppose G is a *minimal vertex guard set* within P: G is a set of vertices of P so that every point of P is seen by at least one point (vertex) of G (i.e., G is a valid vertex guard cover of P), *and* the set G is *minimal*, meaning that deletion of any one point from G will cause G to stop being a valid guard cover of P). Give an example showing that G can have at least 5 times as many points as has a *minimum* vertex guard cover G^* (a set of $g_V(P)$ vertices that is a valid guard cover of P and has the fewest points of any guard cover of P using vertices of P).

(b). How efficiently can one compute a set G of at most $n/2$ vertices of P so that G is a valid guard cover of P?

(c). The following algorithm has been proposed to compute a set G of at most $n/2$ vertices of P so that G is a valid guard cover of P: Walk along the ordered (ccw) list of vertices of P, classifying each as "convex" or "reflex"; let C be the set of convex vertices and let R be the set of reflex vertices; we know that at least one of the sets R or C has at most $n/2$ points – place guards at these points. (You may assume that n is even and that the vertices of P are in general position – no three are collinear.)

(i). How efficient is this algorithm? (in big-Oh)

(ii). Does the algorithm work (to give a valid guard set of at most $n/2$ vertex guards)? If yes, explain briefly why; if no, give a counterexample.

(d). Suppose now that our goal is to find a set of diagonals of P that decompose P into a small number of convex polygons. Let OPT denote the minimum possible number of such convex pieces in a decomposition of P by diagonals. Describe an efficient way to obtain an approximation algorithm for computing OPT. How good is this approximation and what is its running time?

(9). Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\mathcal{F} \subset \mathcal{A}$ and $\mathcal{G} \subset \mathcal{A}$ be two independent σ-algebras on Ω. Suppose a random variable X is measurable both from $(\Omega, \mathcal{F})$ to $(\mathbf{R}, \mathcal{B})$ and from $(\Omega, \mathcal{G})$ to $(\mathbf{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra on the real line $\mathbf{R}$. Show that X is a.s. constant; that is, $P(X = c) = 1$ for some constant c.

(10). Let X_n and X be real-valued random variables in L^2, and suppose that X_n tends to X in L^2. Show that $E\{X_n^2\}$ tends to $E\{X^2\}$.

(11). Let X and Y be two continuous random variables with joint density function $f(x, y) = e^{-y}$ for $0 \leq x \leq y$ and $f(x, y) = 0$ otherwise. Provide an algorithm for generating random variates from $f(x, y)$.

(12). Let $g(x) \geq 0$ be a bounded function over the interval $[0, 1]$. Assume that $g(x)$ has a known upper bound b over $[0, 1]$, i.e., $g(x) \leq b$ for all $x \in [0, 1]$. Consider the following algorithm for estimating the integral $\int_0^1 g(x)dx$:

- generate two independent random numbers $U_1 \sim U(0, 1)$ and $U_2 \sim U(0, 1)$;
- set $X = U_1$ and $Y = bU_2$;
- return $I = \begin{cases} b & \text{if } Y < g(X), \\ 0 & \text{otherwise.} \end{cases}$

1) Is I an unbiased estimator for $\int_0^1 g(x)dx$? Justify your answer.

2) Write down the crude Monte Carlo estimator for approximating $\int_0^1 g(x)dx$. Which estimator do you prefer, the crude Monte Carlo estimator or I?